中国人民大學
RENMIN UNIVERSITY OF CHINA

复杂批次系统智能学习控制

3. Power Update Rule

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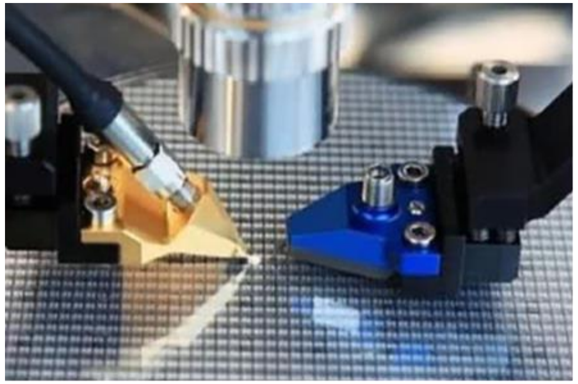
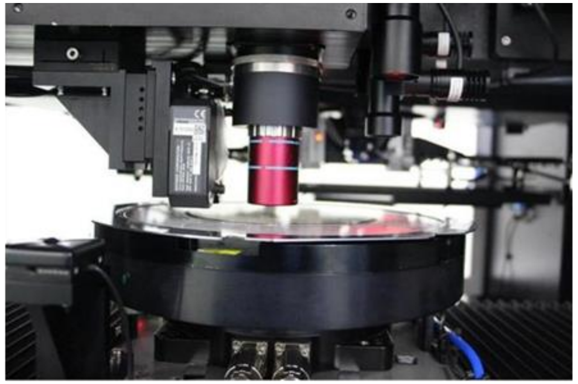
Switching Scheme

融合方案
Fusion Scheme

研究背景




SISO Plant model

$$x_k(t+1) = Ax_k(t) + bu_k(t)$$
$$y_k(t) = cx_k(t)$$

Proportional-type update rule **(PTUR)**

$$u_{k+1}(t) = \underbrace{u_k(t)}_{\text{History input}} + \underbrace{\rho L_t e_k(t+1)}_{\text{Update term}}$$

Tracking error $e_k(t+1)$ $\xrightarrow{\textit{Linearly}\text{ added to}}$ Control input $u_k(t)$

Decreasing $|e_k(t+1)|$

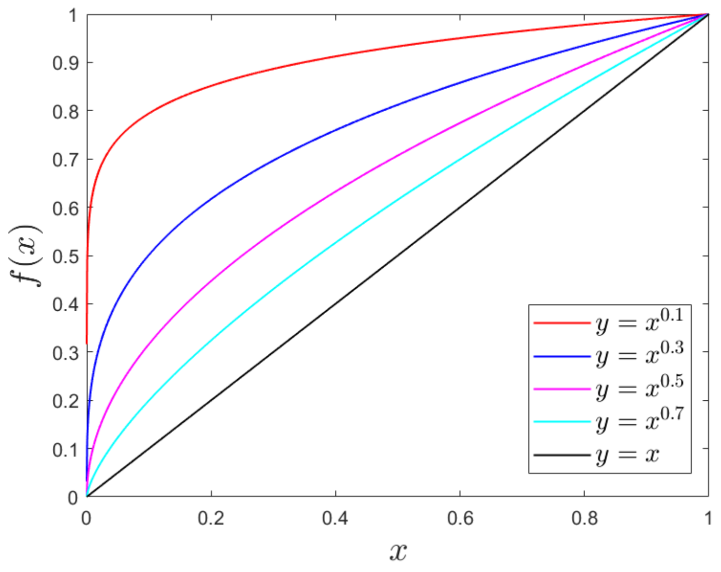
Lower update intensity

Slow convergence rate of $e_k(t+1)$

- How to enhance the learning ability of ILC schemes?
- How to accelerate the convergence speed of ILC schemes?
- How to provide more possibilities to improve the performance of ILC schemes?

Basic question:
How to enhance the update intensity for small error levels?

A fact:
For any number $0 < a < 1$, we have $a^\gamma > a$ with $0 < \gamma < 1$



Fractional-type update rule **(FTUR)**

$$\boxed{u_{k+1}(t) = u_k(t) + \alpha|e_k(t+1)|^\gamma \mathrm{sgn}(e_k(t+1))}$$

$$0 < \gamma < 1$$

Acceleration principle

Fractional power update term ➔ Enhanced update intensity

Convergence analysis

PTUR

$$e_{k+1}(t) = (1 - pcb)e_k(t) - p\sum_{i=1}^{t-1} cA^i b e_k(t-i)$$

FTUR

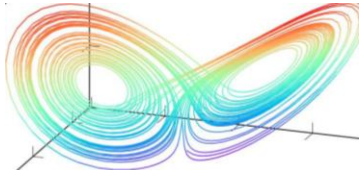
$$e_{k+1}(t) = e_k(t) - \alpha cb|e_k(t)|^\gamma \mathrm{sgn}\big(e_k(t)\big)$$
$$-\alpha \sum_{i=1}^{t-1} cA^i b|e_k(t-i)|^\gamma \mathrm{sgn}(e_k(t-i))$$

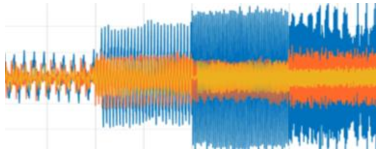
Error dynamics

$$e_{k+1}(t) = e_k(t) - \alpha cb|e_k(t)|^\gamma \operatorname{sgn}(e_k(t)) - \alpha \sum_{i=1}^{t-1} cA^i b|e_k(t-i)|^\gamma \operatorname{sgn}(e_k(t-i))$$

Two main difficulties

Nonlinearity of the error dynamics





Influence of the perturbation term

Difficulty 1: Nonlinearity

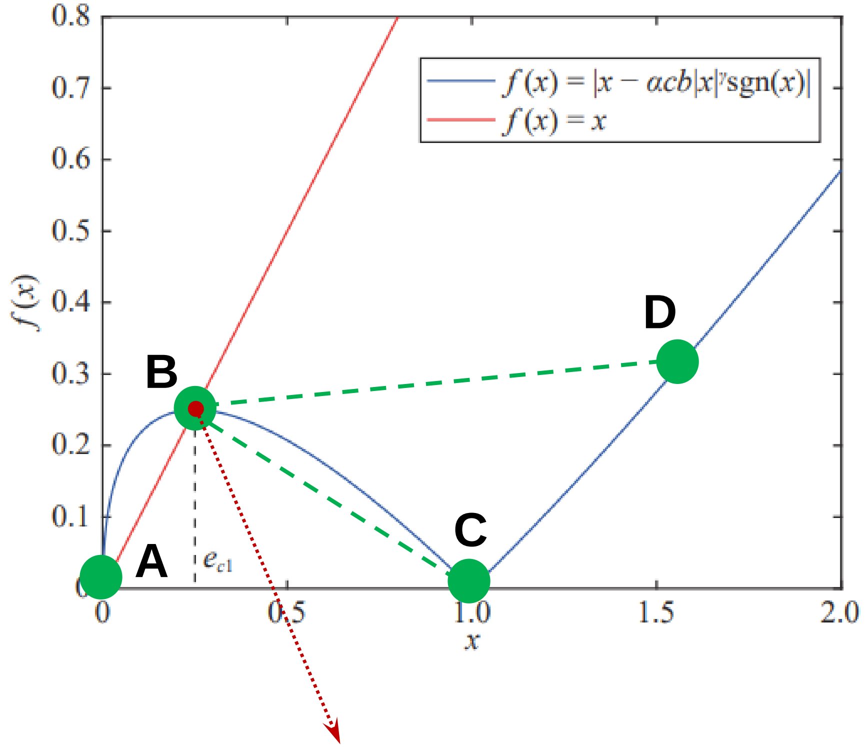
$$x_{k+1} = x_k - \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k)$$

$$|x_{k+1}| = |x_k - \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k)| \triangleq f(|x_k|)$$

Basic convergence analysis theorem: Assume v is the equilibrium of f and B is a ball with a center v. If $\|f(u) - v\| < \|u - v\|$, $\forall u \in B$, $u \neq v$, the iterative sequence generated by $u_{k+1} = f(u_k)$ with any initial value in B eventually converges to v.

a). The equilibrium is an isolated equilibrium point.

b). The distance is decreasing.



Point B is the equilibrium of $f(\cdot)$

The slope of $|BC|$: < 1

The slope of $|BD|$: < 1

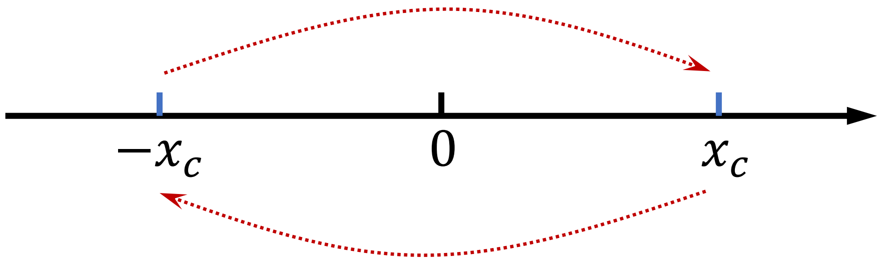
Consistent with the CM principle

The slope of $|AB|$: $= 1$ **Inconsistent** with the CM principle

Convergence to limit cycle

$$|x_{k+1}| = |x_k - \alpha cb|x_k|^{\gamma}\,\mathrm{sgn}(x_k)| \quad + \quad \alpha cb > 0$$

$$x_k \to \pm\left(\frac{\alpha cb}{2}\right)^{\frac{1}{1-\gamma}}$$



Error dynamics

$$e_{k+1}(t) = e_k(t) - \alpha cb|e_k(t)|^\gamma \operatorname{sgn}\big(e_k(t)\big) - \alpha \sum_{i=1}^{t-1} cA^i b|e_k(t-i)|^\gamma \operatorname{sgn}(e_k(t-i))$$

Difficulty 2: Perturbation

$$x_{k+1} = x_k - \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k) - \omega_k, \quad \omega_k \to \pm M_1 \longrightarrow x_{k+1} = -x_k + \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k) - M_1$$

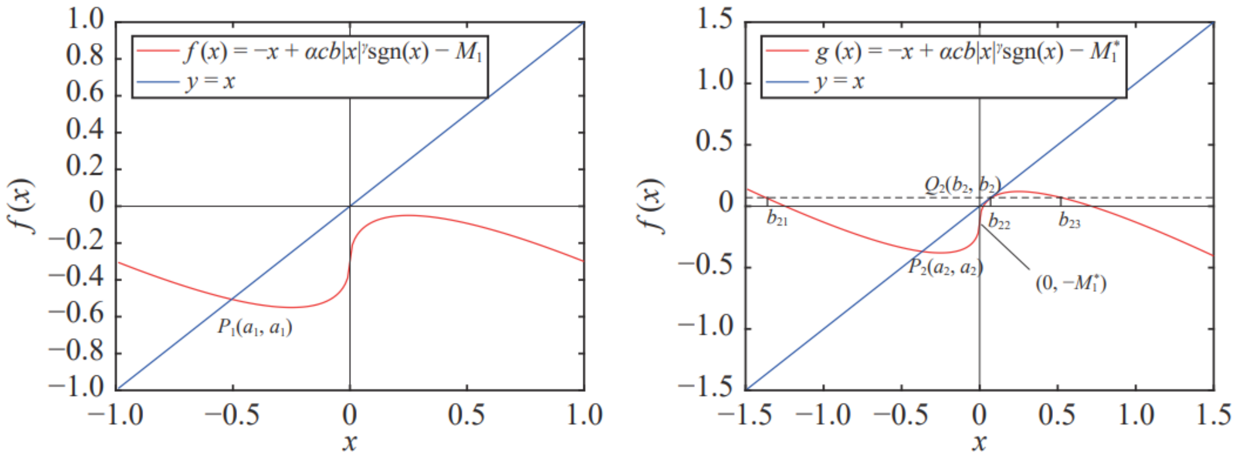
Convergence to a limit cycle ⟵ Convergence to a limit value

Lemma: Suppose $\omega_k \to \pm M_1$. If x_k generated by $x_{k+1} = -x_k + \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k) - M_1$ converges to a, then x_k generated by $x_{k+1} = x_k - \alpha cb|x_k|^\gamma \operatorname{sgn}(x_k) - \omega_k$ converges to the limit cycle $\pm a$.

This lemma establishes the connection between convergence limits and limit cycles!

Convergence

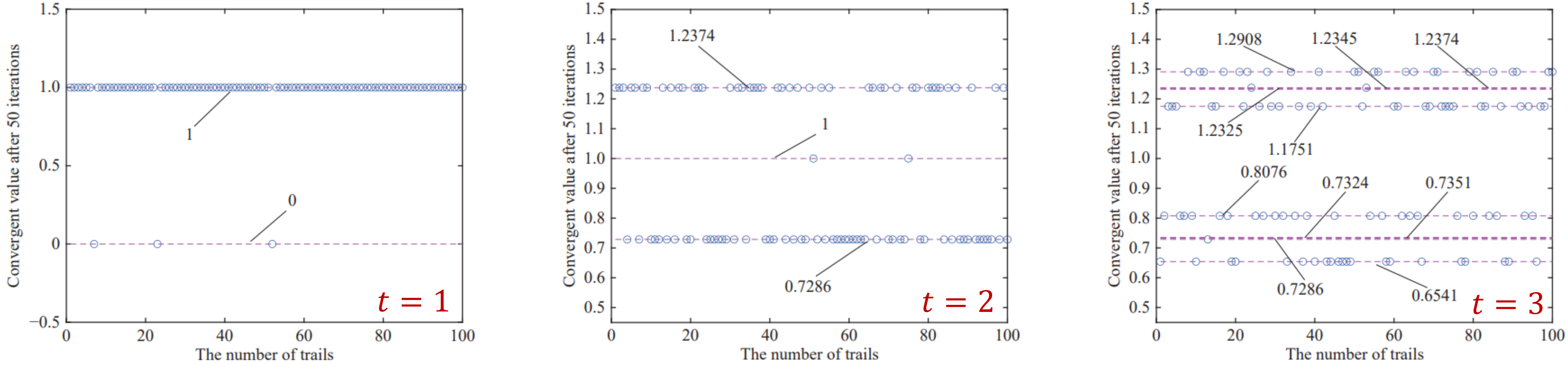
The equation $x = -x + \alpha cb|x|^{\gamma}\,\mathrm{sgn}(x) - M_1$ has one root at least and three roots at most.



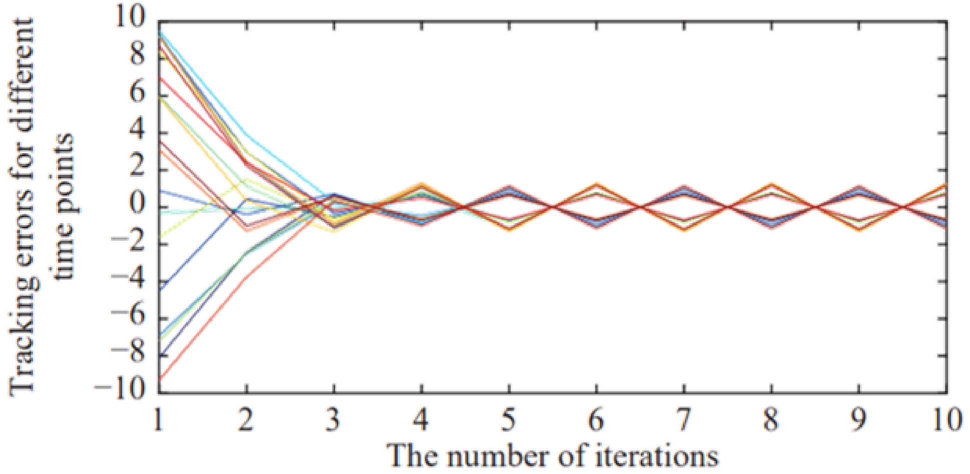
(a) (b) (c)

The roots of $x = -x + \alpha cb|x|^{\gamma}\,\mathrm{sgn}(x) - M_1$ correspond to the equilibriums of $x_{k+1} = -x_k + \alpha cb|x_k|^{\gamma}\,\mathrm{sgn}(x_k) - M_1$.

Theorem: Apply the FTUR to the linear SISO system, the tracking error $e_k(t)$ converges to limit cycles for any t. Further, the limit cycles can be computed by $x = -x + \alpha cb|x|^{\gamma}\,\mathrm{sgn}(x) - M_{t-1}^{\omega}$, where $M_{t-1}^{\omega} = \left|\alpha \sum_{j=1}^{t-1}(-1)^{\omega j} cA^j b e_{c(t-j)}^{\gamma}\right|$.



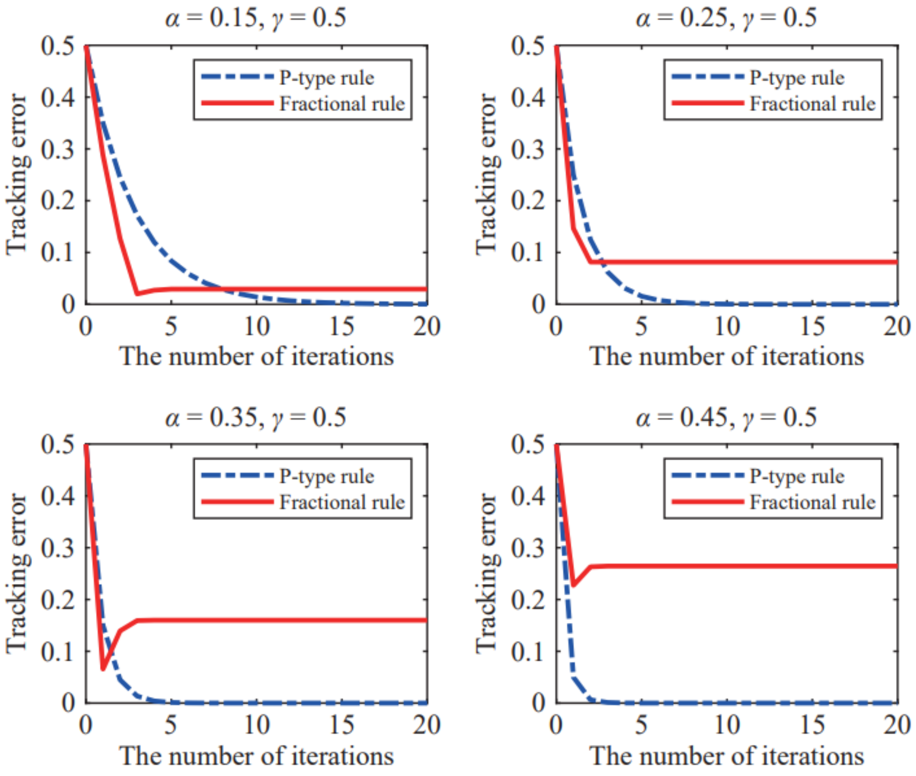
- The convergence values are the roots of the equation $x = -x + \alpha c b |x|^{\gamma}\, \mathrm{sgn}(x) - M_{t-1}^{\omega}$.

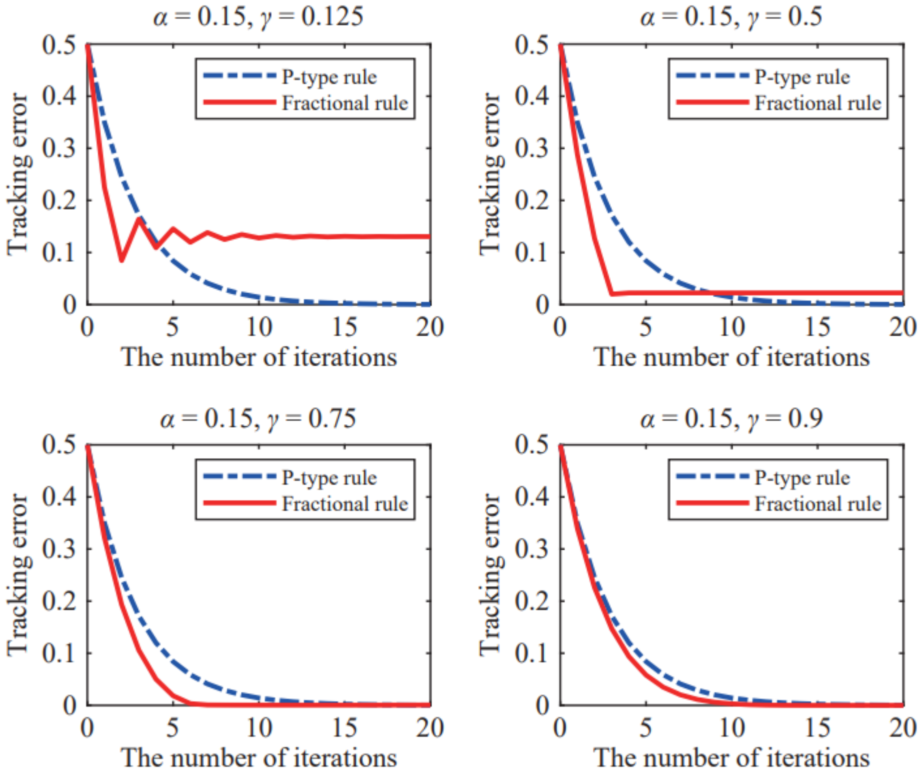


- For each time point, the tracking error converges to a limit cycle.

幂次更新

仿真算例

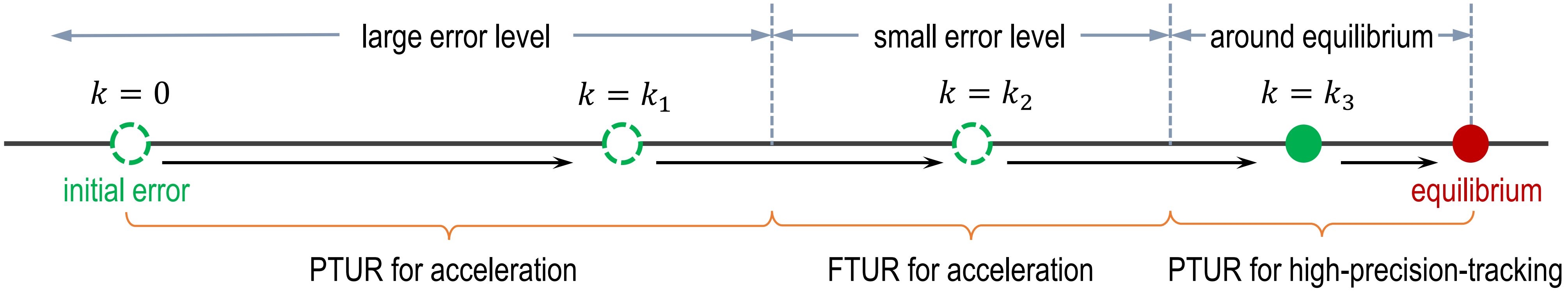




- A small learning gain corresponds to a small convergence limit.
- A small power corresponds to a large convergence limit.
- A small learning gain corresponds to a slow convergence rate.
- A small power corresponds to a fast convergence rate.

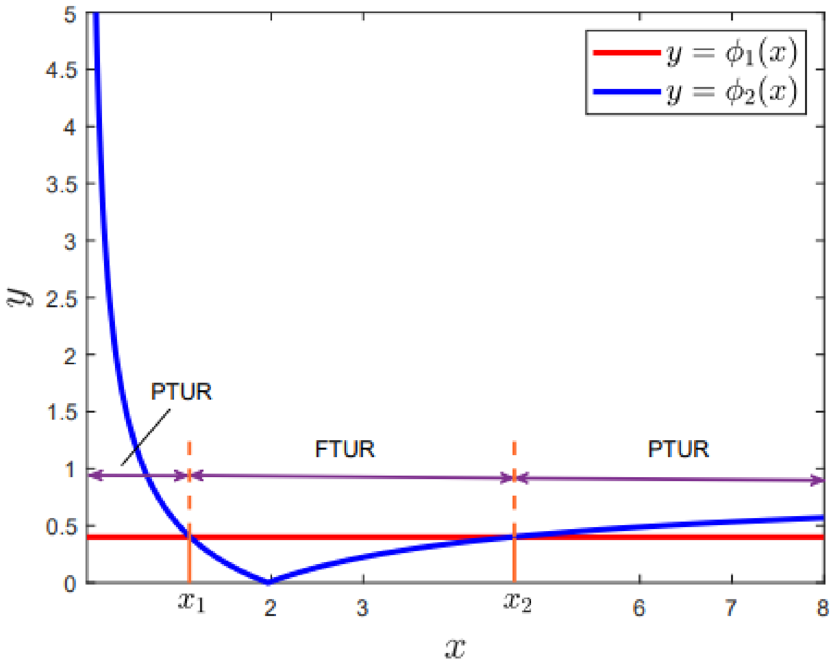
切换方案

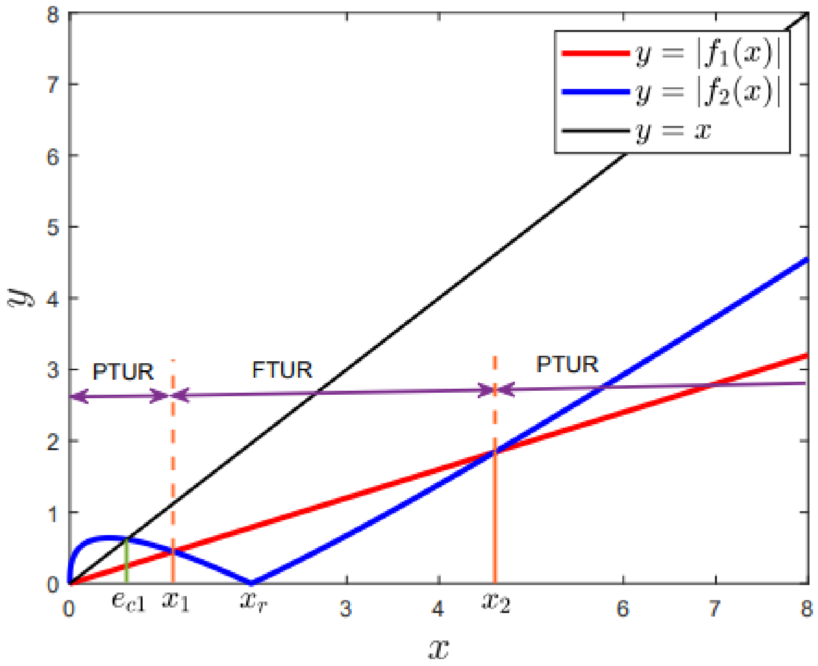
算法设计



MSUR

$$
u_{k+1}(t)=\begin{cases}
u_k(t)+pe_k(t+1), & |e_k(t+1)|>\left(\dfrac{\alpha}{p}\right)^{\frac{1}{1-\gamma}} \quad \textbf{PTUR}\\[2ex]
u_k(t)+\alpha|e_k(t+1)|^{\gamma}\,\mathrm{sgn}\big(e_k(t+1)\big), & \left(\dfrac{\alpha cb}{2-pcb}\right)^{\frac{1}{1-\gamma}}<|e_k(t+1)|\le\left(\dfrac{\alpha}{p}\right)^{\frac{1}{1-\gamma}} \quad \textbf{FTUR}\\[2ex]
u_k(t)+pe_k(t+1), & 0<|e_k(t+1)|<\left(\dfrac{\alpha cb}{2-pcb}\right)^{\frac{1}{1-\gamma}} \quad \textbf{PTUR}
\end{cases}
$$





Two performance advantages

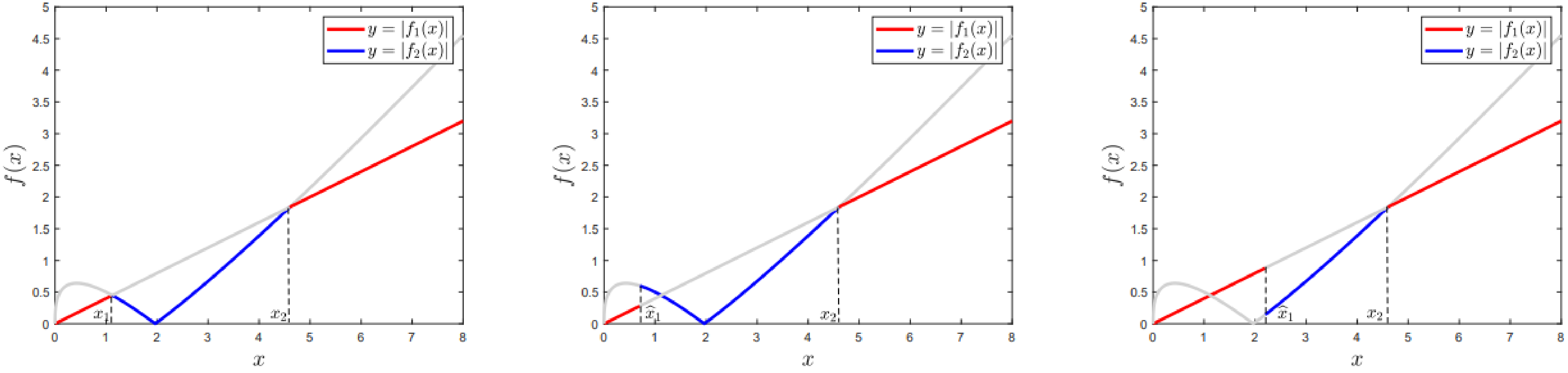
- ✅ Fast convergence for large & small error levels
- ✅ High tracking precision (zero-error convergence)

Nevertheless, MSUR requires the accurate value of cb, which may be hard to obtain for some practical systems

Assumption: There exists $\bar{\theta}$ and $\underline{\theta}$ such that $\underline{\theta} < |cb| < \bar{\theta}$

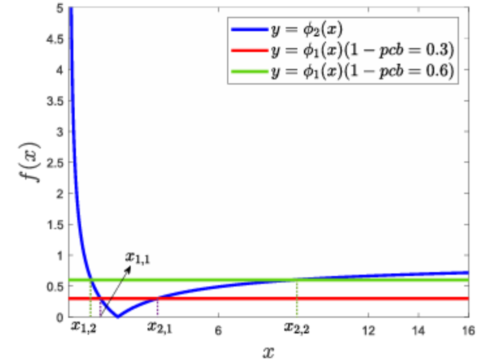
$$u_{k+1}(t) = \begin{cases} u_k(t) + pe_k(t+1), & |e_k(t+1)| > \left(\frac{\alpha}{p}\right)^{\frac{1}{1-\gamma}} \\ u_k(t) + \alpha|e_k(t+1)|^{\gamma}\,\mathrm{sgn}\big(e_k(t+1)\big), & \hat{x}_1 < |e_k(t+1)| \le \left(\frac{\alpha}{p}\right)^{\frac{1}{1-\gamma}} \\ u_k(t) + pe_k(t+1), & 0 < |e_k(t+1)| < \hat{x}_1 \end{cases}$$

$$\max\left\{\left(\frac{|\alpha|\underline{\theta}}{2-p\underline{\theta}}\right)^{\frac{1}{1-\gamma}}, \left|\frac{\alpha\bar{\theta}}{2}\right|^{\frac{1}{1-\gamma}}\right\} < \hat{x}_1 < \left(\frac{|\alpha|\bar{\theta}}{2-p\bar{\theta}}\right)^{\frac{1}{1-\gamma}}$$



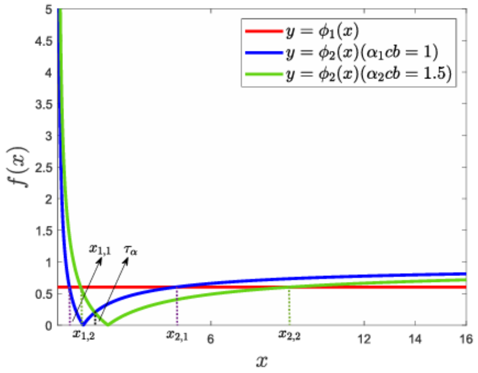
切换方案

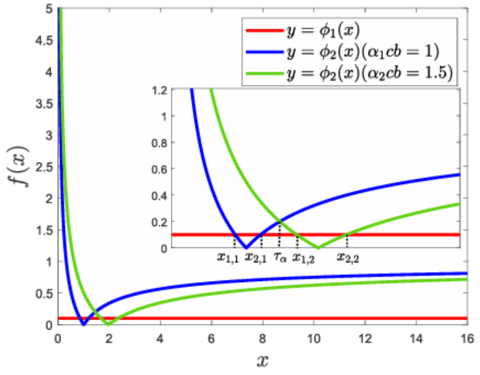
参数影响



a). The effect of parameter p:

A small value of $|1-pcb|$ corresponds to a fast convergence rate.

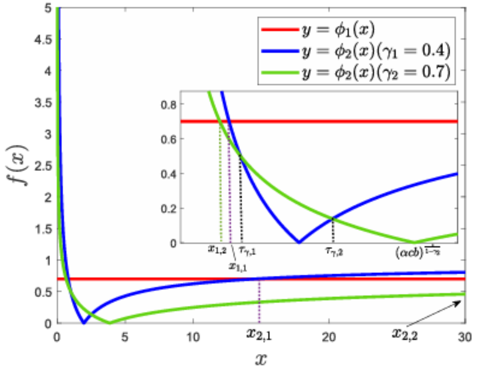


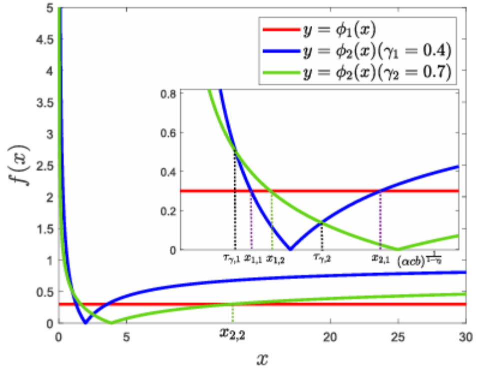


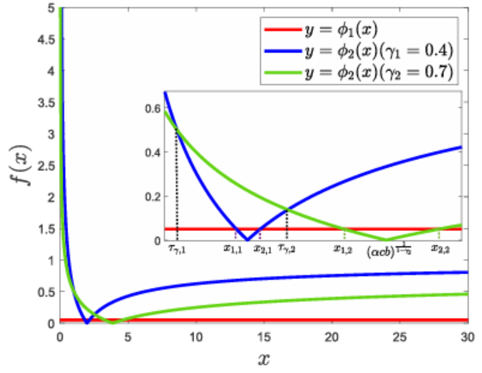
b). The effect of parameter α:

A small value of $|\alpha|$ corresponds to fast convergence for small tracking error levels.

A large value of $|\alpha|$ corresponds to fast convergence for large tracking error levels.



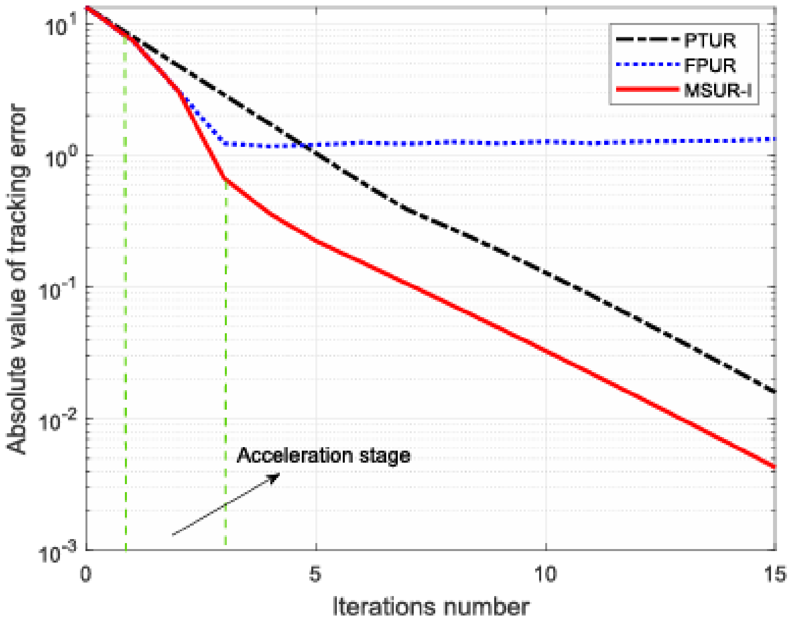


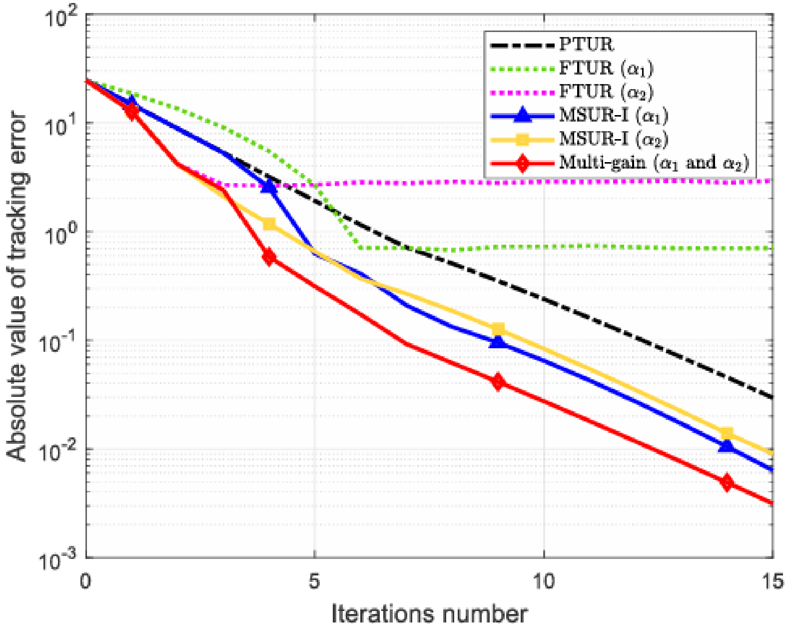


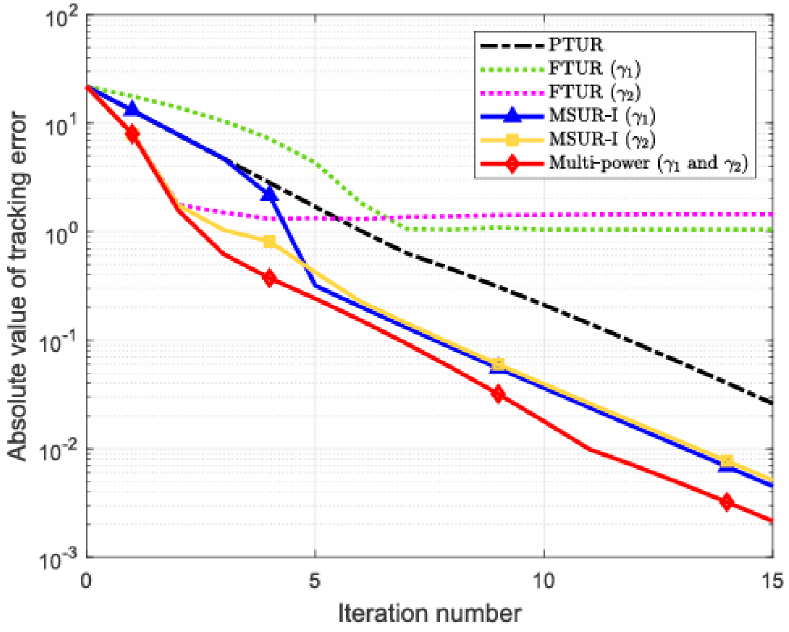
c). The effect of parameter γ:

A small value of $|\gamma|$ corresponds to fast convergence for small tracking error levels.

A large value of $|\gamma|$ corresponds to fast convergence for large tracking error levels.

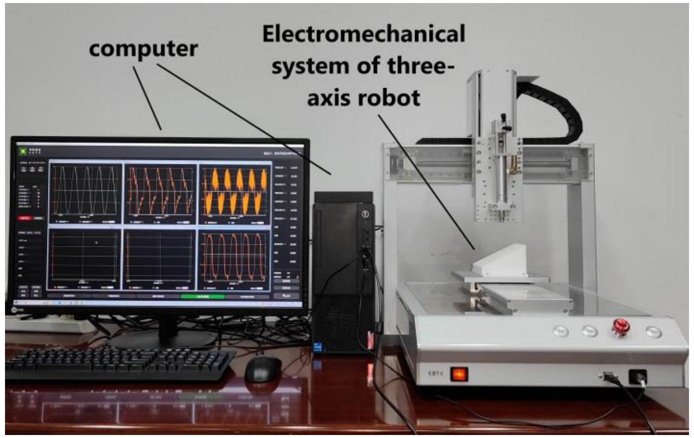


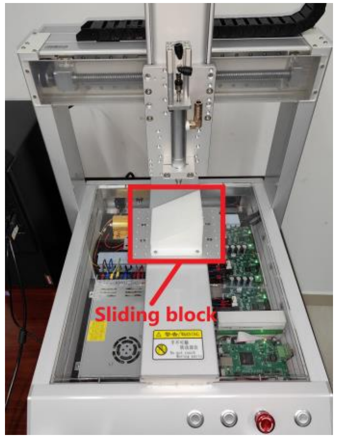




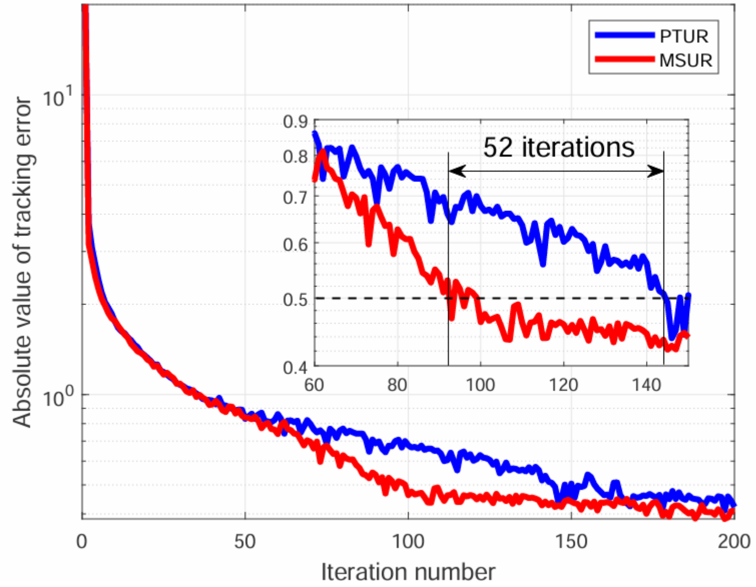
Switching between PTUR and FTUR

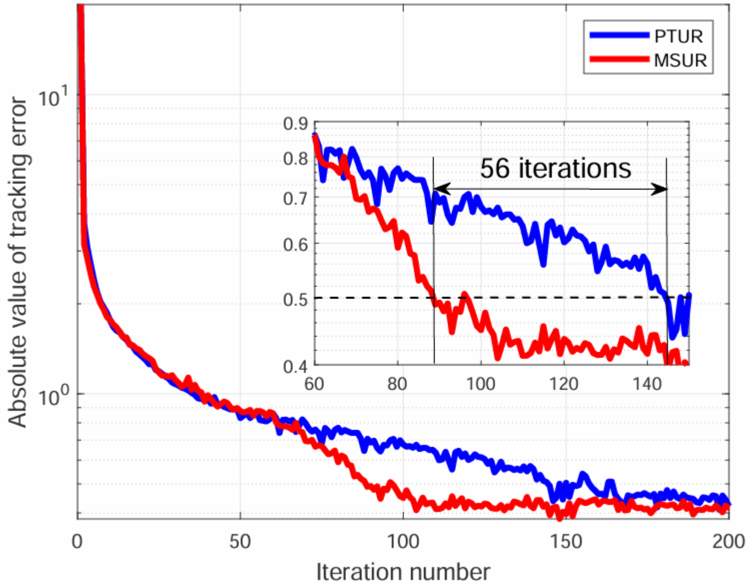
Switching with different α and γ

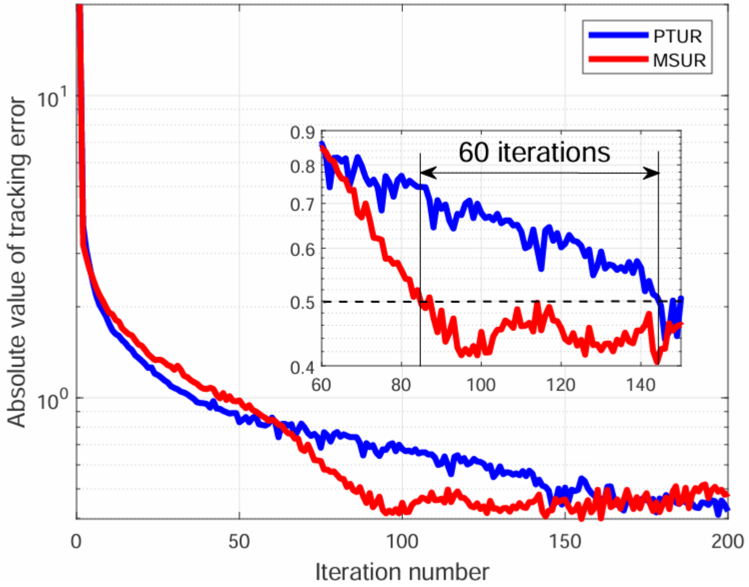




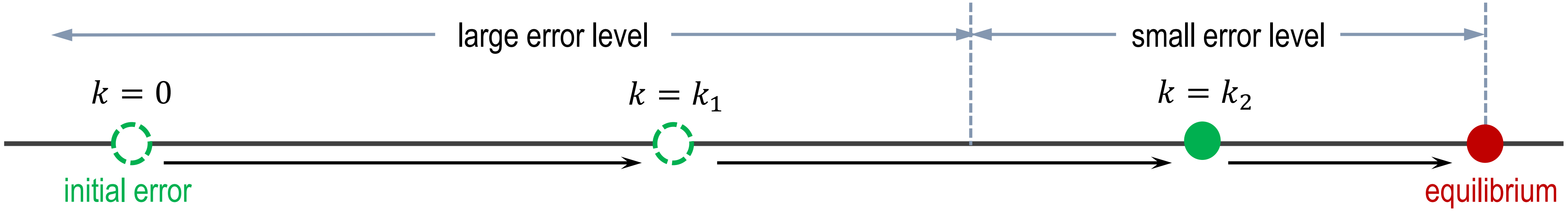
Under the same accuracy requirement, classical PTUR requires nearly **150** iterations, whereas the proposed MSUR requires fewer than **100** iterations.







融合方案 ⊖ 算法设计



Fractional-proportional-type update rule **(FPUR)**

$$\boxed{u_{k+1}(t) = u_k(t) + \alpha |e_k(t+1)|^{\gamma} \operatorname{sgn}\big(e_k(t+1)\big) + \beta e_k(t+1)}$$

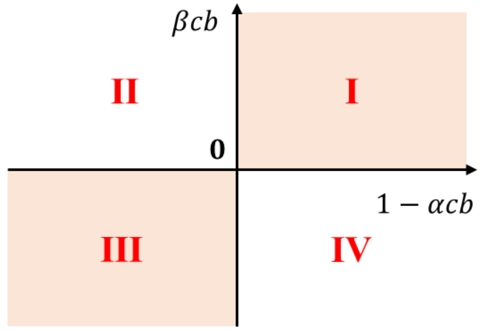
Error dynamics

$$e_{k+1}(t) = (1-\alpha cb) e_k(t) - \beta cb |e_k(t)|^{\gamma} \operatorname{sgn}\big(e_k(t)\big) - \sum_{i=1}^{t-1} cA^i b\big[\alpha e_k(t-i) + \beta |e_k(t-i)|^{\gamma} \operatorname{sgn}\big(e_k(t-i)\big)\big]$$

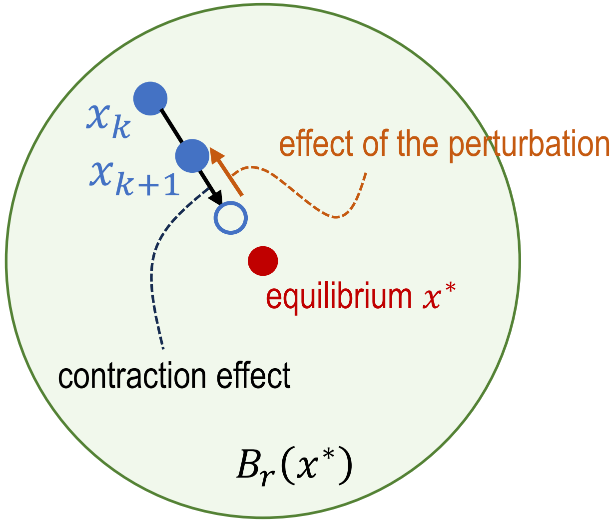
Gain Selection I ⇒ **FPUR-I**

$$\boxed{-1 < 1-\alpha cb < 0, \qquad \beta cb < 0}$$

The terms $1-\alpha cb$ and βcb are selected to have the **same sign**. This selection allows for the cancellation of two terms in the tracking error recursion during the iteration process. These two terms contribute to reducing tracking error in subsequent iterations.



Lemma: Consider the nonlinear iteration with perturbation $x(k+1) = f\big(x(k)\big) + \omega_k$, where $\omega_k \to \omega$ and x^* is the root of $x = f(x) + \omega$. If for every $x \in B_r(x^*)$, there are $\left|\frac{f(x)-f(x^*)}{x-x^*}\right| \le \rho < 1$ and $|\omega_k - \omega| < (1-\rho)r$, then we have $x_k \to x^*$ for any $x(0) = x_0 \in B_r(x^*)$ as the initial value.



This lemma provides an effective way to analyze the convergence of nonlinear perturbed iteration processes.

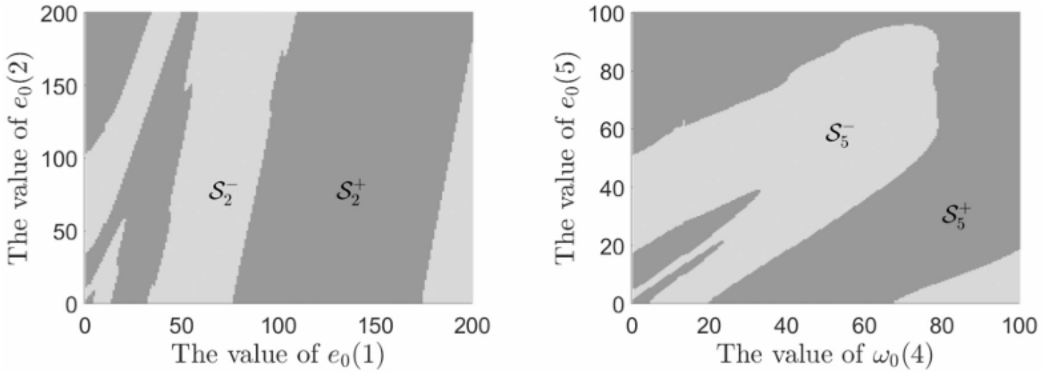
Convergence limit

$$|e_k(t)| \to \left(-\frac{\alpha}{\beta}\right)^{\frac{1}{1-\gamma}}, \forall t \in N^+$$

- The limit of tracking error is **independent** with system matrices

This theorem demonstrates that for any initial value x_0, there exists an open neighborhood of x_0 such that two different initial values correspond to the same convergence limit.

Convergence between limit and initial tracking error



Theorem: According to the convergence limit of tracking errors, the initial values of the system are divided into disjoint open sets.

Convergence rate

If $\gamma = 1 - \frac{1}{\beta cb}$, $e_k(t)$ exhibits a Q-superlinear convergence rate.

If $\gamma \neq 1 - \frac{1}{\beta cb}$, $e_k(t)$ exhibits a Q-linear convergence rate, the convergence rate is described by

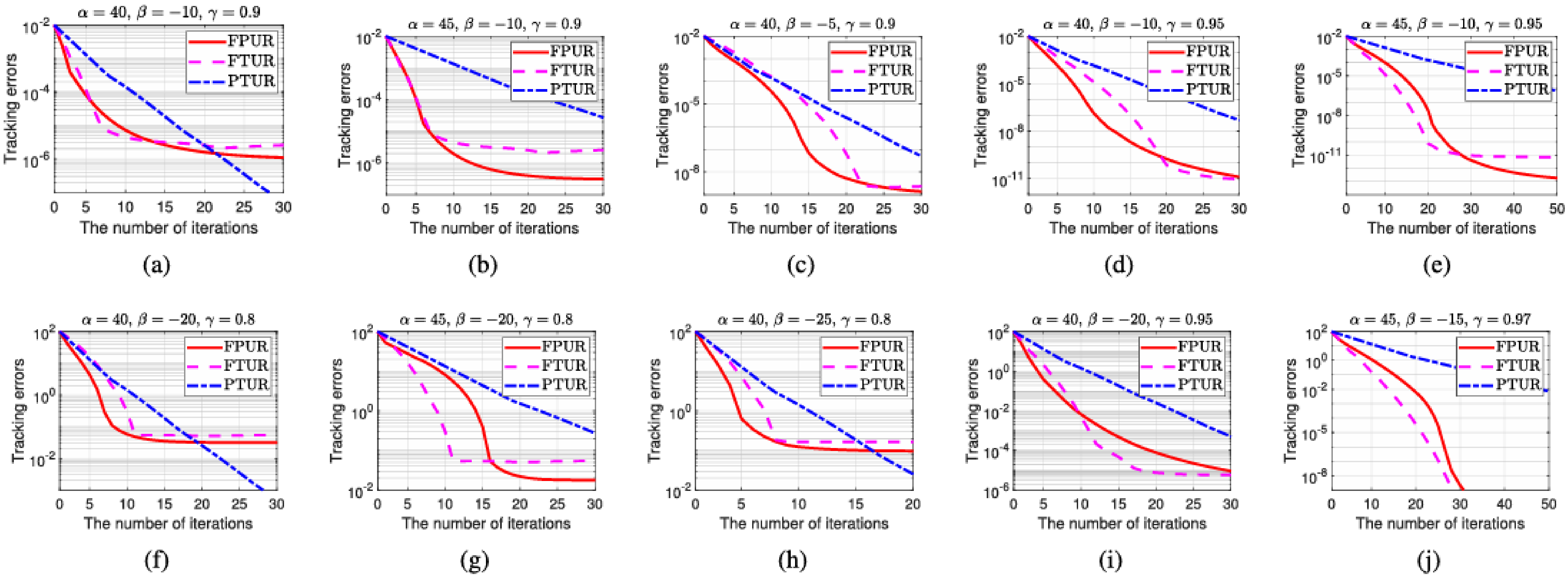
$$\lim_{k\to+\infty} \frac{\|E_{k+1}(t)\|}{\|E_k(t)\|} \leq |1-(1-\gamma)\alpha cb| + \epsilon,$$

where $\epsilon > 0$ is any positive constant.

Parameter selection

High tracking precision: Large $|\alpha|$ and γ, small $|\beta|$

Fast convergence rate: Remains to be explored......


α = 40, β = −10, γ = 0.9
α = 45, β = −10, γ = 0.9
α = 40, β = −5, γ = 0.9
α = 40, β = −10, γ = 0.95
α = 45, β = −10, γ = 0.95
α = 40, β = −20, γ = 0.8
α = 45, β = −20, γ = 0.8
α = 40, β = −25, γ = 0.8
α = 40, β = −20, γ = 0.95
α = 45, β = −15, γ = 0.97
FPUR
FTUR
PTUR
Tracking errors
The number of iterations
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)

Fractional-proportional-type update rule **(FPUR)**

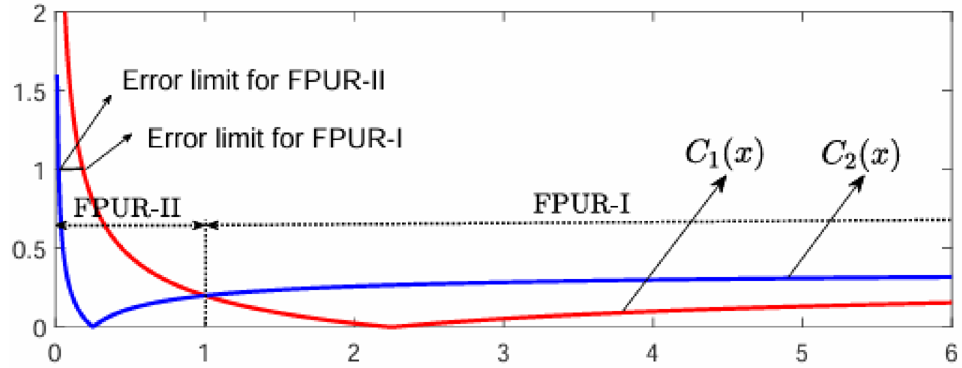
$$u_{k+1}(t) = u_k(t) + \alpha |e_k(t+1)|^{\gamma} \operatorname{sgn}(e_k(t+1)) + \beta e_k(t+1)$$

Gain Selection II ⇒ **FPUR-II**

$$0 < 1 - \alpha c b < 1, \qquad \beta c b > 0$$

FPUR-I $(\bar{\alpha}, \bar{\beta})$ $\quad e_{k+1}(1) = \left[1 - cb\left(\bar{\alpha} + \dfrac{\bar{\beta}}{|e_k(1)|^{\gamma}}\right)\right] e_k(1)$

FPUR-II (α, β) $\quad e_{k+1}(1) = \left[1 - cb\left(\alpha + \dfrac{\beta}{|e_k(1)|^{\gamma}}\right)\right] e_k(1)$



This two gain selections have performance advantages for large and small error levels, respectively.
FPUR-I: fast convergence for large error levels.
FPUR-II: fast convergence for small error levels, and high tracking precision.

Convergence results

Theorem: Apply FPUR-II to the system, we have the following:

(a). If $x = -\varphi(x) + |\omega_{t-1}|$ has one root x_1^*, we have $e_k(t) \to \pm x_1^*$;

(b). If $x = -\varphi(x) + |\omega_{t-1}|$ has two roots x_1^* and x_2^*, we have either $e_k(t) \to \pm x_1^*$ or $e_k(t) \to \pm x_2^*$;

(c). If $x = -\varphi(x) + |\omega_{t-1}|$ has three roots x_1^*, x_2^*, x_3^*, where $x_1^* < x_2^* < x_3^*$, we have either $e_k(t) \to \pm x_1^*$ or $e_k(t) \to \pm x_3^*$.

Here, $\varphi(x) \triangleq (1 - \alpha cb)x - \beta cb|x|^\gamma \mathrm{sgn}(x)$.

Technical lemma for compute limit cycles

Lemma: Consider the following recursion:

$$x_{k+1} = -\rho x_k + \nu |x_k|^\gamma \operatorname{sgn} x_k + \theta_k,$$

where $0 < \rho < 1, \nu > 0$, and $\theta_k \to \theta$, then there exists K such that for $k \geq K$,

$$|x_k| \leq \min\left\{(1-\gamma)\gamma^{\frac{1}{1-\gamma}}\nu^{\frac{1}{1-\gamma}}\rho^{\frac{-\gamma}{1-\gamma}} + |\theta|, \quad \left(\frac{\nu}{1+\rho}\right)^{\frac{1}{1-\gamma}} + \frac{|\theta|}{(1-\gamma)(1-\rho)}\right\}.$$

融合方案 收敛分析

Estimate limit cycles

$$x_{c1} = z_1 \triangleq \left(\frac{\beta cb}{2-\alpha cb}\right)^{\frac{1}{1-\gamma}}$$

$$x_{ct} \le z_t \triangleq \min\left\{ z_1 + \frac{\sum_{i=1}^{t-1}|cA^i b|\left(|\alpha z_{t-i}| + |\beta z_{t-i}^{\gamma}|\right)}{(1-\gamma)(2-\alpha cb)}, \chi(\gamma)\left[\frac{\beta cb}{(1-\alpha cb)^{\gamma}}\right]^{\frac{1}{1-\gamma}} + \sum_{i=1}^{t-1}|cA^i b|\left(|\alpha z_{t-i}| + |\beta z_{t-i}^{\gamma}|\right) \right\}$$

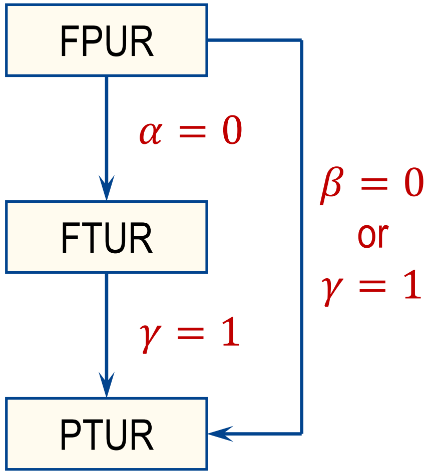
The relationship between the size of the limit cycles and the values of parameters can be obtained from the estimation result.

Relationship between schemes

(a). If $\gamma = 1$ and $0 < (\alpha+\beta)cb < 2$, $z_i = 0$. FPUR-II degenerates to PTUR, $0 < (\alpha+\beta)cb < 2$ is exactly the convergence condition of PTUR.

(b). If $\beta = 0$, we have $z_i = 0$. FPUR-II degenerates to PTUR. $0 < 1 - \alpha cb < 1$ is exactly the convergence condition of PTUR.

(c). If $\alpha = 0$, we have $z_1 = \left(\frac{\beta cb}{2}\right)^{\frac{1}{1-\gamma}}$. FPUR-II degenerates to FTUR.



Convergence rate

If $\gamma = 1 - \frac{1}{2-\alpha cb}$, $e_k(t)$ exhibits a Q-superlinear convergence rate for $t = 1$.

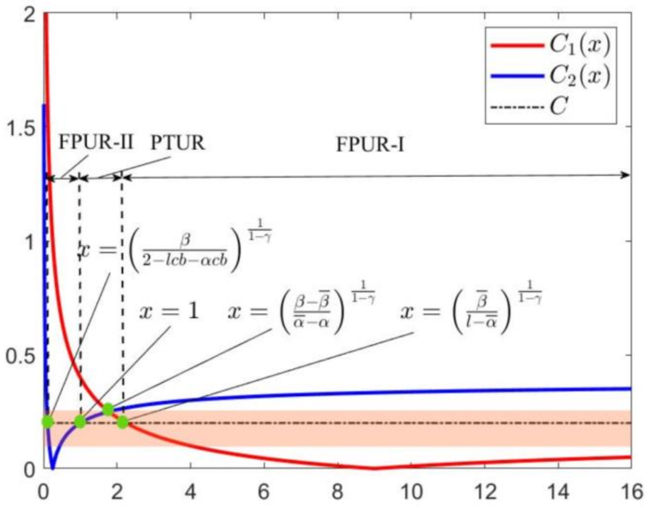
If $\gamma \neq 1 - \frac{1}{2-\alpha cb}$, $e_k(t)$ exhibits a Q-linear convergence rate, the convergence rate is described by $O\left(\ln\frac{1}{\epsilon}\right)$, where $\epsilon > 0$ is the precision requirement.

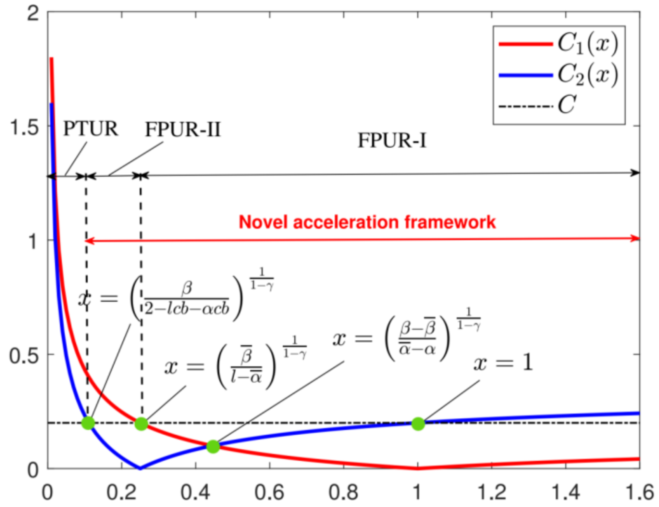
Convergence rate comparison between FPUR-I and FPUR-II

FPUR-I: $$e_{k+1} = \left[1 - cb\left(\bar{\alpha} + \frac{\bar{\beta}}{|e_k|^\gamma}\right)\right]e_k$$

FPUR-II: $$e_{k+1} = \left[1 - cb\left(\alpha + \frac{\beta}{|e_k|^\gamma}\right)\right]e_k$$

Equivalent contraction coefficient





FPUR-I and FPUR-II have performance advantages for large and small error levels, respectively.
The following multistage update scheme is designed to combine the speed advantages of FPUR-I and FPUR-II.

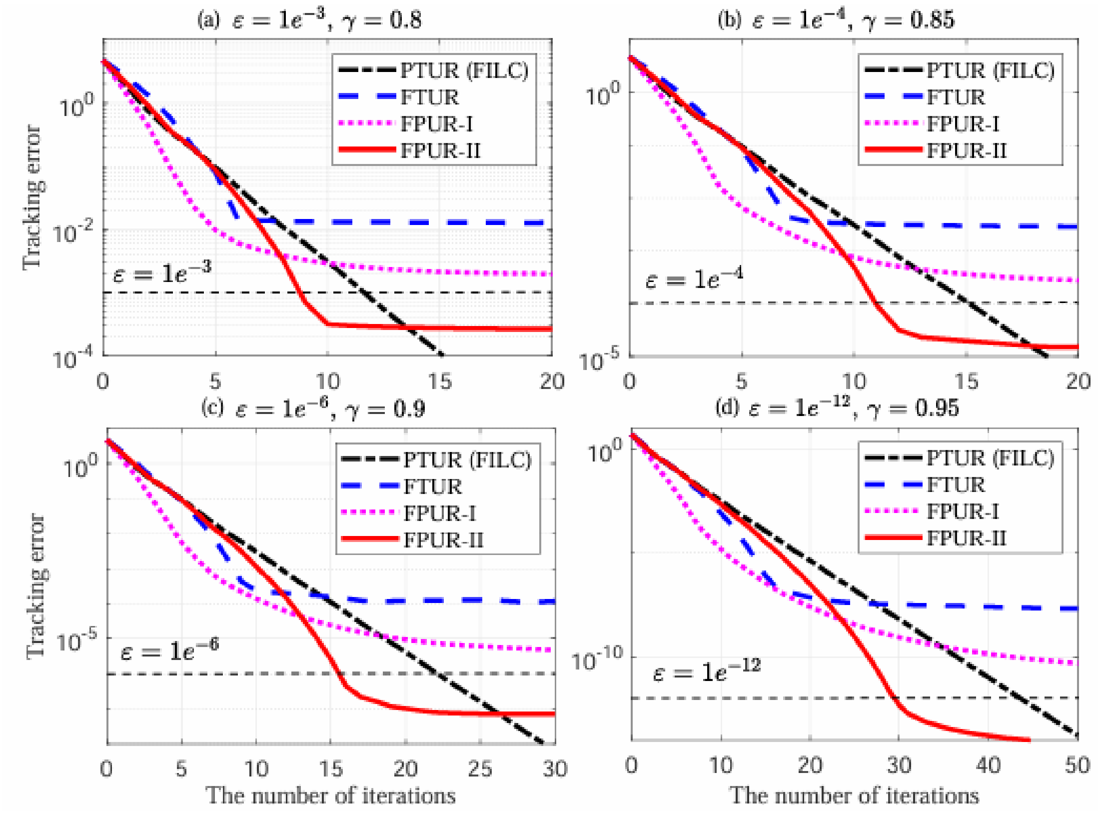
$$u_{k+1}(t)=\begin{cases} u_k(t)+\bar{\beta}e_k(t+1)+\bar{\alpha}|e_k(t+1)|^{\gamma}\operatorname{sgn}\big(e_k(t+1)\big), & |e_k(t)|\geq\left(\dfrac{\beta-\bar{\beta}}{\bar{\alpha}-\alpha}\right)^{\frac{1}{1-\gamma}} \\ u_k(t)+\beta e_k(t+1)+\alpha|e_k(t+1)|^{\gamma}\operatorname{sgn}\big(e_k(t+1)\big), & 0\leq|e_k(t)|\leq\left(\dfrac{\beta-\bar{\beta}}{\bar{\alpha}-\alpha}\right)^{\frac{1}{1-\gamma}} \end{cases}$$

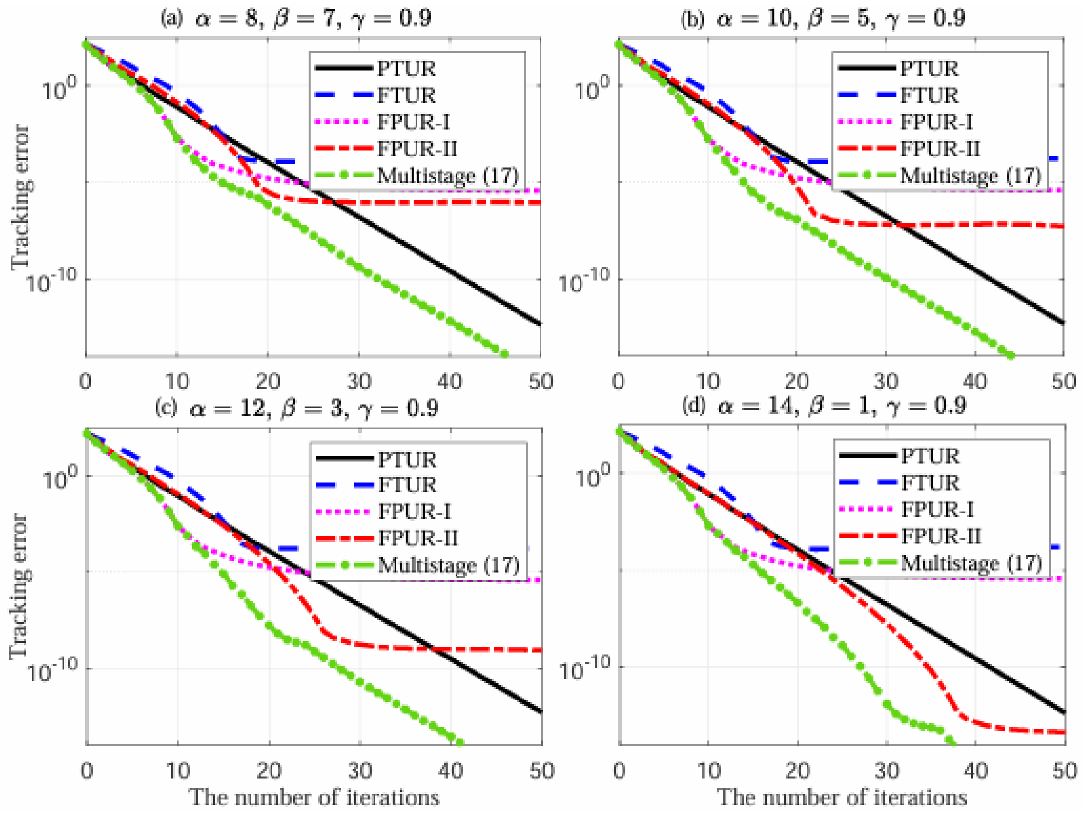
- ✓ Acceleration of the convergence is achieved by using this multistage scheme
- ✓ Without requiring accurate system information in algorithm design

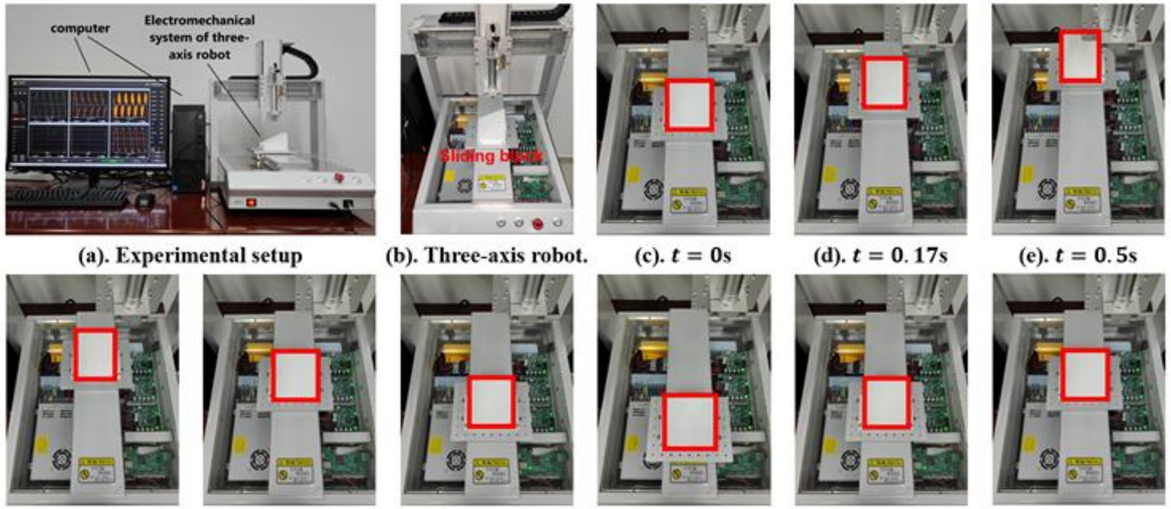
Further, the following multistage update scheme is designed to combine the speed advantages of FPUR-I, FPUR-II, and PTUR.

$$
u_{k+1}(t) = \begin{cases} u_k(t) + \bar{\beta} e_k(t+1) + \bar{\alpha} |e_k(t+1)|^{\gamma} \operatorname{sgn}\big(e_k(t+1)\big) & |e_k(t+1)| \geq \left(\dfrac{\beta - \bar{\beta}}{\bar{\alpha} - \alpha}\right)^{\frac{1}{1-\gamma}} \\ u_k(t) + \beta e_k(t+1) + \alpha |e_k(t+1)|^{\gamma} \operatorname{sgn}\big(e_k(t+1)\big) & \left(\dfrac{\beta cb}{2 - \alpha cb - pcb}\right)^{\frac{1}{1-\gamma}} \leq |e_k(t+1)| \leq \left(\dfrac{\beta - \bar{\beta}}{\bar{\alpha} - \alpha}\right)^{\frac{1}{1-\gamma}} \\ u_k(t) + p e_k(t) & 0 \leq |e_k(t+1)| < \left(\dfrac{\beta cb}{2 - \alpha cb - pcb}\right)^{\frac{1}{1-\gamma}} \end{cases}
$$

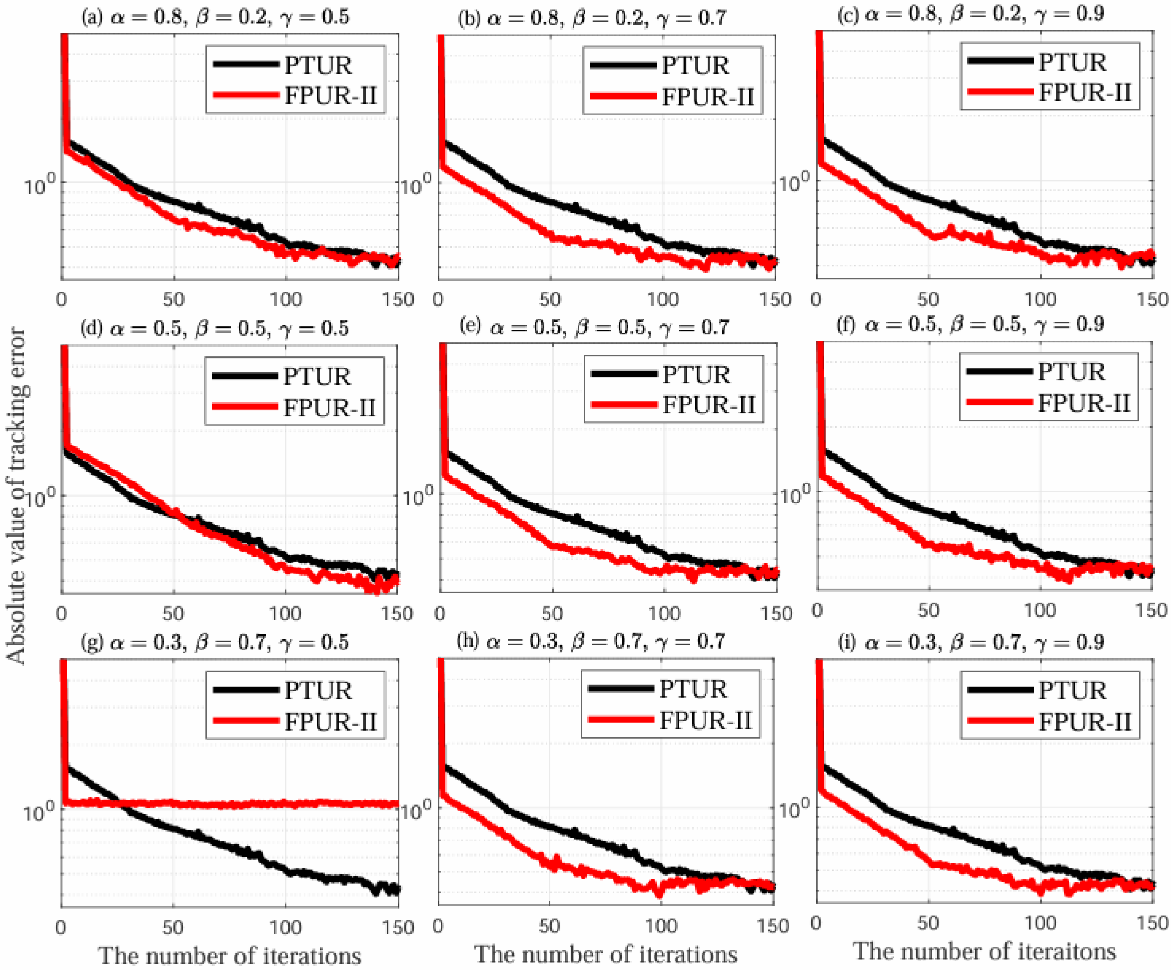
- ✅ Acceleration of the convergence is achieved by using this multistage scheme
- ✅ Zero-error convergence is achieved by using this multistage scheme


(a) $\varepsilon = 1e^{-3}$, $\gamma = 0.8$
(b) $\varepsilon = 1e^{-4}$, $\gamma = 0.85$
(c) $\varepsilon = 1e^{-6}$, $\gamma = 0.9$
(d) $\varepsilon = 1e^{-12}$, $\gamma = 0.95$
PTUR (FILC)
FTUR
FPUR-I
FPUR-II
$\varepsilon = 1e^{-3}$
$\varepsilon = 1e^{-4}$
$\varepsilon = 1e^{-6}$
$\varepsilon = 1e^{-12}$
Tracking error
The number of iterations


(a) $\alpha = 8$, $\beta = 7$, $\gamma = 0.9$
(b) $\alpha = 10$, $\beta = 5$, $\gamma = 0.9$
(c) $\alpha = 12$, $\beta = 3$, $\gamma = 0.9$
(d) $\alpha = 14$, $\beta = 1$, $\gamma = 0.9$
PTUR
FTUR
FPUR-I
FPUR-II
Multistage (17)
Tracking error
The number of iterations



(a). Experimental setup (b). Three-axis robot. (c). $t = 0$s (d). $t = 0.17$s (e). $t = 0.5$s
(f). $t = 0.83$s (g). $t = 1$s (h). $t = 1.17$s (i). $t = 1.5$s (j). $t = 1.83$s (k). $t = 2$s



Under the same accuracy requirement, FPUR-II requires nearly **50** fewer iterations than PTUR.

Thanks

http://shendong.tech/

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